On a Conjecture of E. Dittert

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Dedicated to Professor Mun-Gu Sohn on his 60th birthday

Submitted by Richard A. Brualdi

ABSTRACT

Let K_n denote the set of all $n \times n$ nonnegative matrices whose entries have sum n, and let ϕ be a real valued function defined on K_n by $\phi(X) = \prod_{i=1}^n r_i + \prod_{j=1}^n c_j - \operatorname{per} X$ for $X \in K_n$ with row sum vector $(r_1, \dots, r_n)$ and column sum vector $(c_1, \dots, c_n)$. For the same X, let $\phi_{ij}(X) = \prod_{k \neq i} r_k + \prod_{l \neq j} c_l - \operatorname{per} X(i|j)$. $A \in K_n$ is called a ϕ-maximizing matrix if $\phi(A) \geqslant \phi(X)$ for all $X \in K_n$. Dittert's conjecture asserts that $J_n = [1/n]_{n \times n}$ is the unique ϕ-maximizing matrix on K_n. In this paper, the following are proved: (i) If $A = [a_{ij}]$ is a ϕ-maximizing matrix on K_n, then $\phi_{ij}(A) = \phi(A)$ if $a_{ij} > 0$, and $\phi_{ij}(A) \leqslant \phi(A)$ if $a_{ij} = 0$. (ii) The conjecture is true for $n = 3$.

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1. INTRODUCTION

Throughout this paper, let K_n denote the set of all $n \times n$ real nonnegative matrices whose entries have sum n, and let ϕ denote a real valued function defined on K_n by

$$\phi(X) = \prod_{i=1}^{n} \sum_{j=1}^{n} x_{ij} + \prod_{j=1}^{n} \sum_{i=1}^{n} x_{ij} - \operatorname{per} X$$

for $X = [x_{ij}] \in K_n$, where per stands for the permanent function. A matrix $A \in K_n$ is called a *ϕ-maximizing matrix* on K_n if $\phi(A) \geqslant \phi(X)$ for all $X \in K_n$. Let Ω_n denote the set of all $n \times n$ doubly stochastic matrices, and

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let J_n denote the $n \times n$ matrix all of whose entries are $1/n$. An $n \times n$ matrix is called *fully indecomposable* if it does not contain an $s \times t$ zero submatrix with $s + t = n$. As usual, let E_{ij} denote a (0,1) matrix of suitable size all of whose entries are 0 except for the (i, j) entry, which is 1. For a matrix A, let $A(i|j)$ denote the matrix obtained from A by deleting row i and column j. We write $A > 0$ ($A \not> 0$) to denote that A is a positive (not a positive) matrix.

For the function ϕ on K_n, there is a conjecture which is apparently due to E. Dittert.

CONJECTURE [3, Conjecture 28]. J_n is the unique ϕ-maximizing matrix on K_n.

R. Sinkhorn [4] proved that every ϕ-maximizing matrix on K_n has a positive permanent and also that the conjecture is true for $n = 2$. In a recent paper [1], the present author investigated some properties of ϕ-maximizing matrices and proved the validity of the conjecture for positive semidefinite symmetric matrices in K_n and for matrices in a sufficiently small neighborhood of J_n in K_n.

For a matrix A, let R_A and C_A denote the row sum vector and the column sum vector of A respectively. Let $A \in K_n$, and let $R_A = (r_1, \dots, r_n)$, $C_A = (c_1, \dots, c_n)$. For $1 \leqslant i, j \leqslant n$, we define $\phi_{ij}(A)$ by

$$\phi_{ij}(A) = r_1 \cdots r_{i-1} r_{i+1} \cdots r_n + c_1 \cdots c_{j-1} c_{j+1} \cdots c_n - \operatorname{per} A(i|j).$$

In this paper, we prove that if $A = [a_{ij}]$ is a ϕ-maximizing matrix on K_n, then

$$\phi_{ij}(A) = \phi(A) \quad \text{if} \quad a_{ij} > 0,$$
$$\phi_{ij}(A) \leqslant \phi(A) \quad \text{if} \quad a_{ij} = 0$$

and prove the conjecture for $n = 3$.

2. ϕ-MAXIMIZING MATRICES

The following lemma can be proved by using one of the results in [1]. But here we give a short proof.

LEMMA 1. *Let $A = [a_{ij}]$ be a ϕ-maximizing matrix on K_n. Then*

(i) $\phi_{ij}(A) = \phi_{kl}(A)$ *if* $a_{ij} > 0$ *and* $a_{kl} > 0$,
(ii) $\phi_{ij}(A) \leqslant \phi_{kl}(A)$ *if* $a_{ij} = 0$ *and* $a_{kl} > 0$.

Proof. Let ε be a real number with sufficiently small absolute value, and let $A_\varepsilon = A + \varepsilon(E_{ij} - E_{kl})$. Then $A_\varepsilon \in K_n$ and

$$\phi(A_\varepsilon) = \phi(A) + \varepsilon\big[\phi_{ij}(A) - \phi_{kl}(A)\big] + O(\varepsilon^2).$$

Thus $\phi_{ij}(A) = \phi_{kl}(A)$, and assertion (i) is proved.

For (ii), if we use a positive ε, then $A_\varepsilon \in K_n$ and it must be that $\phi_{ij}(A) - \phi_{kl}(A) \leqslant 0$. ■

From Lemma 1 we get the following

THEOREM 1. *Let $A = [a_{ij}]$ be a ϕ-maximizing matrix on K_n. Then*

(i) $\phi_{ij}(A) = \phi(A)$ *if* $a_{ij} > 0$,
(ii) $\phi_{ij}(A) \leqslant \phi(A)$ *if* $a_{ij} = 0$.

Proof. Let $R_A = (r_1,\dots,r_n)$, $C_A = (c_1,\dots,c_n)$, $r = r_1 \cdots r_n$, $c = c_1 \cdots c_n$, and $\bar{r}_i = r/r_i$, $\bar{c}_j = c/c_j$ for $i, j = 1,\dots,n$. Suppose $a_{ij} > 0$. Then for (k,l) with $a_{kl} > 0$, we have $\phi_{ij}(A) = \phi_{kl}(A)$, which implies

$$a_{kl}\phi_{ij}(A) = a_{kl}\phi_{kl}(A) \qquad (k,l = 1,\dots,n),$$

that is,

$$a_{kl}\phi_{ij}(A) = a_{kl}\bar{r}_k + a_{kl}\bar{c}_l - a_{kl}\operatorname{per} A(k|l) \qquad (k,l=1,\dots,n).$$

Taking summation over k, l, we get

$$\begin{aligned} n\phi_{ij}(A) &= \sum_{k=1}^{n}\sum_{l=1}^{n} a_{kl}\bar{r}_k + \sum_{k=1}^{n}\sum_{l=1}^{n} a_{kl}\bar{c}_l - \sum_{k=1}^{n}\sum_{l=1}^{n} a_{kl}\operatorname{per} A(k|l) \\ &= \sum_{k=1}^{n} r_k\bar{r}_k + \sum_{l=1}^{n} c_l\bar{c}_l - n\operatorname{per} A \\ &= \sum_{k=1}^{n} r + \sum_{l=1}^{n} c - n\operatorname{per} A \\ &= n(r + c - \operatorname{per} A) \\ &= n\phi(A), \end{aligned}$$

which completes the proof of (i). Assertion (ii) can be proved by a similar computation by using (ii) of Lemma 1. ∎

We note that if, in Theorem 1, the matrix A is doubly stochastic, then the assertions $\phi_{ij}(A)=\phi(A)$ and $\operatorname{per} A(i|j)=\operatorname{per} A$ are the same, and $\phi_{ij}(A) \leqslant \phi(A)$ and $\operatorname{per} A(i|j) \geqslant \operatorname{per} A$ are the same.

LEMMA 2 [1, Lemma 2]. *Let $A=[a_{ij}]$ be a ϕ-maximizing matrix on K_n, and let $1 \leqslant s < t \leqslant n$. If $\phi_{is}(A)=\phi_{it}(A)$ for all i such that $a_{is}+a_{it}>0$, then the matrix obtained from A by replacing each of the columns s and t by its average is also a ϕ-maximizing matrix on K_n. A similar statement holds for rows.*

LEMMA 3 [1, Theorem 1]. *If A is a positive ϕ-maximizing matrix on K_n, then $A=J_n$.*

THEOREM 2. *If $\phi_{ij}(A)=\phi(A)$ for all $1 \leqslant i, j \leqslant n$ and for every ϕ-maximizing matrix A on K_n, then J_n is the unique ϕ-maximizing matrix on K_n.*

Proof. Assume that $\phi_{ij}(A)=\phi(A)$ for all $i, j=1,\dots,n$ for every ϕ-maximizing matrix A on K_n. Let A be a ϕ-maximizing matrix on K_n. Suppose $A \notin \Omega_n$. Let $C_A=(c_1,\dots,c_n)$. Then, without loss of generality, we may assume that $0<c_1 \leqslant c_2 \leqslant \cdots \leqslant c_n$ with $c_1<1$. Let

$$M_n=(J_2 \oplus I_{n-2})(I_1 \oplus J_2 \oplus I_{n-3}) \cdots (I_{n-3} \oplus J_2 \oplus I_1)(I_{n-2} \oplus J_2).$$

Then $M_n A$ is a ϕ-maximizing matrix on K_n by Lemma 2 applied several times. So $B=\lim_{k \to \infty} M_n^k A$ is a ϕ-maximizing matrix. But B is a positive ϕ-maximizing matrix which is not even a doubly stochastic matrix, since its first column has sum $c_1<1$, contradicting Lemma 3. Thus we have shown that $A \in \Omega_n$. It now follows that $A=J_n$ from the van der Waerden–Egoryčev theorem. ∎

3. DITTERT'S CONJECTURE FOR $n=3$

One hopes to prove that if ϕ is maximal at $A \in K_n$, then $A=J_n$. It seems that one would like to show here that even if $A \neq J_n$, at the very least, A is fully indecomposable. But at the present time I am not able to prove it except for the case $n=3$.

LEMMA 4 [1]. *Let A be a ϕ-maximizing matrix on K_n, and let $1 \leqslant s < t \leqslant n$. If columns s and t of A have either the same sums or the same $(0,1)$ patterns, then the matrix obtained from A by replacing each of columns s and t by its average is also a ϕ-maximizing matrix on K_n. A similar statement holds for rows.*

The following lemma is a special case of a lemma in [2].

LEMMA 5 [2]. *Suppose that A and B are ϕ-maximizing matrices on K_n with the same $(0,1)$ pattern. If $\phi(tA+(1-t)B)$ is a polynomial in t of degree $\leqslant 3$, then $\phi(tA+(1-t)B)$ is a constant function of t.*

Now we are ready to prove

LEMMA 6. *If A is a ϕ-maximizing matrix on K_3, then A is fully indecomposable.*

Proof. Let $A=[a_{ij}]$ be a ϕ-maximizing matrix on K_3, and let $R_A=(r_1,r_2,r_3)$, $C_A=(c_1,c_2,c_3)$. Assume that A is not fully indecomposable. Then, without loss of generality, we may assume that $a_{12}=a_{13}=0$. Then $a_{11}>0$. First we are to show that $(a_{21},a_{31})\neq(0,0)$. Suppose that $a_{21}=a_{31}=0$. Let $B=A(1|1)$ and let $a_{11}=a$. Then $A=aI_1\oplus B$, and thus $\phi(A)=a\phi(B)$ and $\phi_{11}(A)=\phi(B)$. Now by Theorem 1, $a=1$ and hence $B\in K_2$. So, by Sinkhorn [4], $B=J_2$. But then $\phi(A)=\phi(B)=\phi(J_2)<\phi(J_3)$, a contradiction. Thus it must be that $(a_{21},a_{31})\neq(0,0)$.

Now, suppose, on the other hand, that $a_{21}>0$ and $a_{31}>0$. Then, by Lemma 1, $\phi_{21}(A)=\phi_{31}(A)$ implies that $0=\phi_{21}(A)-\phi_{31}(A)=r_1r_3-r_1r_2=r_1(r_3-r_2)$, i.e. that $r_2=r_3$. Thus, by Lemma 4, we may assume that

$$A=(I_1\oplus J_2)A=\begin{bmatrix} a & 0 & 0\\ x & y & y\\ x & y & y\end{bmatrix},$$

where $axy>0$. Suppose $x\geqslant y$. Then

$$A_1=\begin{bmatrix} a & 0 & 0\\ x+y & y & 0\\ x-y & y & 2y\end{bmatrix}$$

is also a ϕ-maximizing matrix, since $R_{A_1}=R_A$, $C_{A_1}=C_A$, and $\operatorname{per}A_1=\operatorname{per}A$.

Now, $\phi_{23}(A_1) - \phi_{33}(A_1) = ay > 0$, contradicting Lemma 1. Thus $x < y$. Let

$$A_2 = \begin{bmatrix} a & 0 & 0 \\ 2x & y & y-x \\ 0 & y & y+x \end{bmatrix}.$$

Then A_2 is a ϕ-maximizing matrix on K_3 and hence, by Lemma 4,

$$A_3 = \begin{bmatrix} a & 0 & 0 \\ 2x & y-x/2 & y-x/2 \\ 0 & y+x/2 & y+x/2 \end{bmatrix}$$

is also a ϕ-maximizing matrix on K_3. But, since $R_{A_2} = R_{A_3}$, $C_{A_2} = C_{A_3}$, and $\operatorname{per} A_3 = 2a(y^2 - x^2/4) < 2ay^2 = \operatorname{per} A_2$, we have $\phi(A_3) > \phi(A_2) = \phi(A)$, a contradiction. Thus it must be that exactly one of a_{21} or a_{31} is 0. So, without loss of generality, we may assume that $a_{21} > 0$ and $a_{31} = 0$, and hence that A has the form

$$A = \begin{bmatrix} a & 0 \quad 0 \\ b & \\ 0 & \;\; B \end{bmatrix}$$

with $ab > 0$.

We claim that B is not a positive matrix. For, suppose $B > 0$. Then, by Lemma 4, we may say

$$A = \begin{bmatrix} a & 0 & 0 \\ b & x & x \\ 0 & y & y \end{bmatrix}$$

with $xy > 0$. Now, by Lemma 1, we have $0 = \phi_{11}(A) - \phi_{21}(A) = 2y(b + x - a)$, i.e. $a = b + x$. For a sufficiently small $\varepsilon > 0$, let

$$A_\varepsilon = \begin{bmatrix} a-\varepsilon & \varepsilon & 0 \\ b+\varepsilon & x-\varepsilon & x \\ 0 & y & y \end{bmatrix}.$$

Then $R_{A_\varepsilon} = R_A$, $C_{A_\varepsilon} = C_A$, and

$$\begin{aligned} \operatorname{per} A_\varepsilon &= \operatorname{per} A + \varepsilon y(b - 2x - a) + O(\varepsilon^2) \\ &= \operatorname{per} A - 3\varepsilon xy + O(\varepsilon^2) < \operatorname{per} A, \end{aligned}$$

which gives us $\phi(A_\varepsilon) > \phi(A)$, a contradiction. Therefore it must be that $B \not> 0$.

Now, since $a \operatorname{per} B = \operatorname{per} A > 0$, we have $\operatorname{per} B > 0$, and hence we may assume that

$$B = \begin{bmatrix} x & u \\ v & y \end{bmatrix} \quad \text{with} \quad xy > 0 \text{ and } uv = 0.$$

If $v = 0$, then, for a sufficiently small $\varepsilon > 0$,

$$\begin{bmatrix} a & 0 & 0 \\ b-\varepsilon & x & u+\varepsilon \\ \varepsilon & 0 & y-\varepsilon \end{bmatrix}$$

is a matrix in K_3 whose ϕ-value is strictly greater than $\phi(A)$. Thus $v > 0$ and $u = 0$, and hence A finally has the form

$$A = \begin{bmatrix} a & 0 & 0 \\ b & x & 0 \\ 0 & v & y \end{bmatrix}$$

with $abxyv > 0$. Let

$$A_1 = \begin{bmatrix} y & 0 & 0 \\ v & x & 0 \\ 0 & b & a \end{bmatrix}.$$

Then A_1 is also a ϕ-maximizing matrix on K_3 with the same $(0,1)$ pattern as A. Since $\phi(tA + (1-t)A_1)$ is a polynomial in t of degree $\leqslant 3$,

$$\tfrac{1}{2}(A + A_1) = \begin{bmatrix} \frac{1}{2}(a+y) & 0 & 0 \\ \frac{1}{2}(b+v) & x & 0 \\ 0 & \frac{1}{2}(b+v) & \frac{1}{2}(a+y) \end{bmatrix}$$

is also a ϕ-maximizing matrix on K_3, by Lemma 5. By renaming $\frac{1}{2}(a+y)$ and $\frac{1}{2}(b+v)$ as a and b respectively in $\frac{1}{2}(A + A_1)$, we may assume that

$$A = \begin{bmatrix} a & 0 & 0 \\ b & x & 0 \\ 0 & b & a \end{bmatrix}.$$

Now, Lemma 1 gives us that $0 = \phi_{22}(A) - \phi_{21}(A) = -ax < 0$, a contradiction, and the proof is complete. ■

THEOREM 3. *J_3 is the unique ϕ-maximizing matrix on K_3.*

Proof. Let $A = [a_{ij}]$ be a ϕ-maximizing matrix on K_3. If we can show that $A > 0$, then $A = J_3$ by Lemma 3 and we will be done. Suppose $A \ngtr 0$. Since A is fully indecomposable, there are three possibilities for the $(0,1)$ pattern of A up to permutations of rows and columns, namely (i) a_{11} is the only zero entry of A; (ii) a_{11}, a_{22} are only zero entries of A; (iii) a_{11}, a_{22}, a_{33} are only zero entries of A.

Case (i). By Lemmas 4 and 5, we may assume that

$$A = \begin{bmatrix} 0 & x & x \\ x & y & y \\ x & y & y \end{bmatrix}$$

with $xy > 0$. Lemma 1 gives us that $0 = \phi_{13}(A) - \phi_{23}(A) = 2y(2y - x)$, i.e. that $x = 2y$, which yields

$$A = \frac{1}{4}\begin{bmatrix} 0 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

and $\phi(A) < \phi(J_3)$, a contradiction.

Case (ii). By Lemma 5, we may say that A has the form

$$A = \begin{bmatrix} 0 & x & y \\ x & 0 & z \\ y & z & u \end{bmatrix}$$

with $xyzu > 0$. Let

$$A_1 = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus I_1\right) A \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus I_1\right)$$

$$= \begin{bmatrix} 0 & x & z \\ x & 0 & y \\ z & y & u \end{bmatrix}.$$

Then A_1 is also a ϕ-maximizing matrix on K_3 with the same (0,1) pattern as A. So, by Lemma 5, we may assume that $y = z$. Then, since the columns 1 and 2 have the same column sums $x + y$, $A_2 = A(J_2 \oplus I_1)$ is also a ϕ-maximizing matrix by Lemma 4. Now, since $A_2 > 0$, it must be that $A_2 = J_3$ by Lemma 3. Thus it follows that $x = \frac{2}{3}$, $y = z = u = \frac{1}{3}$, and hence that $\phi(A) = 2 - \frac{8}{27} < \phi(J_3)$, a contradiction.

Case (iii). By Lemma 5 again, we may assume that A is of the form

$$A = \begin{bmatrix} 0 & x & y \\ x & 0 & x \\ y & x & 0 \end{bmatrix}.$$

Since the columns 1 and 3 have the same column sum, by Lemma 4 we can get a ϕ-maximizing matrix with exactly one zero entry by averaging these two columns, which is shown to be impossible in case (i).

Thus it has been proved that $A > 0$, and the proof is complete. ■

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